2.4 (Burden) Error Analysis for Iterative Methods.

Def. – 1) $p_n \to p$

2) $p_n \neq p$, for all n.

If there exists $\alpha, \lambda > 0$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

We will say p_n converges to p of order α.

If $\alpha = 1 \longrightarrow$ linear convergence.

$\alpha = 2 \longrightarrow$ quadratic convergence

$\alpha = n \longrightarrow$ Convergence of order n.

Exercise (4) (5) $p_n = \frac{1}{n^2}$ Converges linearly to zero.

Proof. –

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^2} - 0\right|}{\left|\frac{1}{n^2} - 0\right|} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1.$$

$p_n = \frac{1}{n}$, also converges to zero linearly

In general,

$p_n = \frac{1}{n^k}$, $k = 1, 2, 3, \ldots$ converges to zero linearly.

Example:

a) $$p_n = \frac{1}{10^{2^n}}$$

$$p_n \to p = 0$$

and $$\frac{|p_{n+1} - p|}{|p_n - p|} = \frac{10^{2^n}}{10^{2^{n+1}}} = \frac{10^{2^n}}{10^{2^n * 2}} = \frac{10^{2^n}}{\left(10^{2^n}\right)^2} = \frac{1}{10^{2^n}}$$

Also, $$\frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{\left(10^{2^n}\right)^2}{10^{2^{n+1}}} = \frac{10^{2^{n+1}}}{10^{2^{n+1}}} = 1$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = 1$$

$$\Rightarrow \quad p_n = \frac{1}{10^{2^n}} \xrightarrow[n \to \infty]{} 0$$ quadratically

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b) $$p_n = \frac{1}{10^{2^{2n}}}, \qquad p_n \to p = 0$$

$$\frac{|p_{n+1} - 0|}{|p_n - 0|} = \frac{10^{2^{2n}}}{10^{2^{2n+2}}} = \frac{10^{2^{2n}}}{10^{2^{2n} * 2^2}} = \frac{10^{2^{2n}}}{\left(10^{2^{2n}}\right)^4}$$

$\Rightarrow \quad p_n \to p$ of order 4.

Relative speed of convergence.

n	$p_n = \frac{1}{n}$	$p_n = \frac{1}{n^2}$	$p_n = \frac{1}{10^{2^n}}$
1	1	1	1
2	5×10^{-1}	2.5×10^{-1}	10^{-4}
3	3.3×10^{-1}	1.1×10^{-1}	10^{-8}
4	2.5×10^{-1}	6.25×10^{-2}	10^{-16}
5	2×10^{-1}	4×10^{-2}	9.9×10^{-33}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	1.1×10^{-1}	1.23×10^{-2}	9.9×10^{-512}
10	10^{-1}	10^{-2}	10^{-1024}
	linear	linear	quadratic.

Quadratic Convergence of Newton Method.

Consider the equation

$$\boxed{f(x) = 0,} \qquad (3.1)$$

where $f \in C^3[a,b]$ and there exists $p \in (a,b)$ such that

$$\boxed{f(p) = 0} \qquad (3.2)$$

Also,

$$\boxed{f'(p) \neq 0} \qquad (3.3)$$

Quadratic
Convergence of Newton method

Assuming: $f \in C^3[a,b]$, $p \in [a,b]$
$f(p) = 0$, $f'(p) \neq 0$

From thm 2.6, the functional iteration

$$p_{n+1} = g(p_n) = p_n - \frac{f(p_n)}{f'(p_n)} \rightarrow p$$

Of $\quad g(x) = x - \dfrac{f(x)}{f'(x)}$

Converges to p in an interval $[p-\delta, p+\delta] \subset [a,b]$.

Also, for $x \in [p-\delta, p+\delta]$, the first order Taylor polynomial of $g(x)$ is given by

$$g(x) = g(p) + (x-p)\, g'(p) + \frac{(x-p)^2}{2}\, g''(\xi(x))$$

with $\xi(x)$ btw x and p.

Now, since

$$g'(x) = 1 - \frac{f'(x) f'(x) - f(x) f''(x)}{f'^2(x)} = \frac{f(x) f''(x)}{f'^2(x)}$$

$$g'(p) = \frac{\overset{=0}{f(p)}\, f''(p)}{\underset{\neq 0}{f'(p)}} = 0$$

Thus,

$$g(x) = g(p) + \frac{(x-p)^2}{2} g''(\xi(x))$$

and

$$p_{n+1} = g(p_n) = p + \frac{(p_n-p)^2}{2} g''(\xi_n), \quad \xi_n \text{ btw } p_n \text{ and } p.$$

Then, $$|p_{n+1} - p| = \frac{1}{2} |p_n - p|^2 |g''(\xi_n)|$$

and $$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{1}{2} \lim_{n\to\infty} |g''(\xi_n)|$$

Now, $$g''(x) = \frac{(f')^3 f'' + f(f')^2 f''' - 2 f f' (f'')^2}{f'^4(x)}$$

Since $f'(x) \neq 0$ in $[p-\delta_1, p+\delta_1] \supset [p-\delta, p+\delta]$

and f', f'', f''' are conts. in $[a,b] \supset [p-\delta, p+\delta]$

then, $g''(x)$ is continous in $[p-\delta, p+\delta]$

Therefore

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{1}{2} |g''(p)| = \frac{1}{2} \left| \frac{f''(p)}{f'(p)} \right| = \frac{1}{2} M$$

As a consequence $p_n \to p$ of order 2 if $g''(p) \neq 0$
of higher order if $g''(p) = 0$,
where $|g''(x)| < M$ in $[p-\delta, p+\delta]$

Zeros of functions and multiplicity

Def.— A root p of $f(x)=0$ is a zero of multiplicity m of $f(x)$ if for $x \neq p$, we can write
$$f(x) = (x-p)^m q(x), \text{ where}$$
$$\lim_{x \to p} q(x) \neq 0.$$

Theorem.— (Simple zeros or roots with multiplicity one).
$f \in C^1[a,b]$ has a simple zero p in $[a,b]$ if and only if $f(p)=0$, but $f'(p) \neq 0$.

Proof.— ($\longrightarrow$) p a simple zero of f implies (def)
$f(p)=0$ and $f(x) = (x-p)\,q(x)$ where
$$\lim_{x \to p} q(x) \neq 0.$$

Now
$$f'(p) \overset{\text{Hip.}}{=} \lim_{x \to p} f'(x) = \lim_{x \to p} \left[q(x) + (x-p)\,q'(x) \right]$$
$$= \lim_{x \to p} q(x) \neq 0$$

$(\Leftarrow)$ Since $f \in C'[a,b]$ then it can be written as a Taylor polynomial of degree zero about p.

$$f(x) = f(p) + (x-p)\, f'(\xi(x))$$

where $\xi(x)$ is between x and p.

Since $f(p) = 0 \Rightarrow f(x) = (x-p)\, f'(\xi(x))$

$$= (x-p)\,(f' \circ \xi)(x)$$

$$= (x-p)\, q(x)$$

and clearly, $\lim\limits_{x \to p} q(x) = \lim\limits_{x \to p} f'(\xi(x))$

$$= f'\left(\lim_{x \to p} \xi(x)\right) = f'(p) \neq 0$$

~~21~~ 10

$f \in C^m[a,b]$

p is a zero of multipl. $m \iff 0 = f(p) = f'(p) = \dots = f^{(m-1)}(p)$

but $f^{(m)}(p) \neq 0$.

Proof.-

$(\rightarrow)$ p is a zero of multiplicity m. if

$$f(x) = (x-p)^m q(x)$$

where $\lim_{x \to p} q(x) \neq 0$

~~By induction. It's true for $m \in \mathbb{N}$~~

$$f'(x) = m(x-p)^{m-1} q(x) + (x-p)^m q'(x)$$

$$\lim_{x \to p} f'(x) = 0.$$

$$f''(x) = m(m-1)(x-p)^{m-2} q(x) + \underline{m(x-p)^{m-1} q'(x)}$$
$$+ \underline{m(x-p)^{m-1}} q'(x) + (x-p)^m q''(x)$$

$$f''(x) = m(m-1)(x-p)^{m-2} + 2m(x-p)^{m-1} q'(x) +$$
$$+ (x-p)^m q''(x)$$

con from here. !

Exercise (1a) Sect 2.4 Burden.

Use Newton to find solutions accurate to within 10^{-5} to the following problem

$$f(x) = x^2 - 2xe^{-x} + e^{-2x} = 0, \qquad 0 \leq x \leq 1$$

In order to apply Newton, we need $f'(x)$.

$$f'(x) = 2x - 2e^{-x} + 2xe^{-x} - 2e^{-2x} =$$

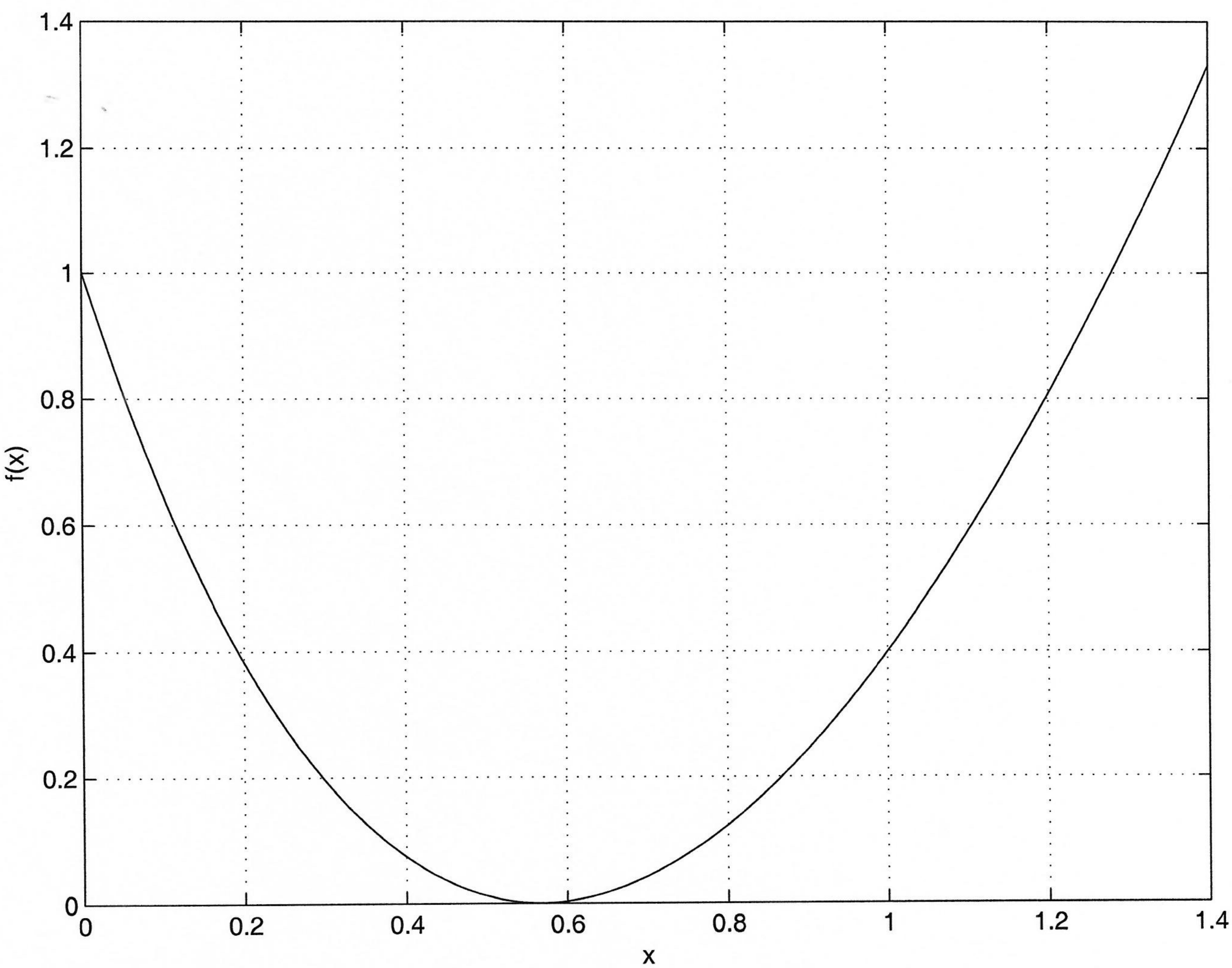
$$= 2\left[-e^{-2x} + xe^{-x} - e^{-x} + x\right]$$

So functional iteration

$$\boxed{p_n = p_{n-1} - \frac{p_{n-1}^2 - 2p_{n-1}e^{-p_{n-1}} + e^{-2p_{n-1}}}{2\left[-e^{-2p_{n-1}} + p_{n-1}e^{-p_{n-1}} - e^{-p_{n-1}} + p_{n-1}\right]}}$$

As we will see in the execution of Newton algorithm for this function, Newton is converging slow. Not of Quadratic order.

12'



12²

```
>>    Input data
     A              B           TOL           itmax          po
              0  1.4000e+000  1.0000e-005  2.0000e+002  2.5000e-001

Exiting fzero: aborting search for an interval containing a sign change
    because NaN or Inf function value encountered during search
(Function value at -463.1595 is Inf)
Check function or try again with a different starting value.
MATLAB Approx.
   NaN
```

it	approx.	ErrBound	f(pn)	AbsErr
1.0000e+000	3.9864e-001	1.4864e-001	7.4307e-002	NaN
2.0000e+000	4.8019e-001	8.1554e-002	1.9174e-002	NaN
3.0000e+000	5.2297e-001	4.2773e-002	4.8710e-003	NaN
4.0000e+000	5.4488e-001	2.1909e-002	1.2276e-003	NaN
5.0000e+000	5.5596e-001	1.1088e-002	3.0815e-004	NaN
6.0000e+000	5.6154e-001	5.5780e-003	7.7194e-005	NaN
7.0000e+000	5.6434e-001	2.7975e-003	1.9318e-005	NaN
8.0000e+000	5.6574e-001	1.4009e-003	4.8320e-006	NaN
9.0000e+000	5.6644e-001	7.0098e-004	1.2083e-006	NaN
1.0000e+001	5.6679e-001	3.5062e-004	3.0212e-007	NaN
1.1000e+001	5.6697e-001	1.7534e-004	7.5534e-008	NaN
1.2000e+001	5.6706e-001	8.7681e-005	1.8884e-008	NaN
1.3000e+001	5.6710e-001	4.3842e-005	4.7211e-009	NaN
1.4000e+001	5.6712e-001	2.1922e-005	1.1803e-009	NaN
1.5000e+001	5.6713e-001	1.0961e-005	2.9507e-010	NaN
1.6000e+001	5.6714e-001	5.4805e-006	7.3768e-011	NaN

```
 Approximated Root and Number of Iterations
     5.671378098536595e-001    1.600000000000000e+001

 >>
```

The problem is that $f(x)$ has a zero at p of multiplicity higher than 1 inside the interval $[0,1]$. Therefore, $f'(p)=0$, this is why Newt is converging slower than quadratic.

A <u>way to overcome this difficulty</u> is to define a new function $\underline{\underline{\mu(x)}}$ from $f(x)$ that has p as a root, but p be a root of multiplicity one. ($\mu'(p) \neq 0$).

<u>Definition</u>:

$$\mu(x) \equiv \frac{f(x)}{f'(x)}, \quad x \neq p \quad \mu(p) = \frac{f(p)}{f'(p)} = \frac{0}{0} \quad ?$$

However, if p is a zero of multiplicity m

$$f(x) = (x-p)^m q(x), \quad \lim_{x \to p} q(x) \neq 0.$$

Thus, for $\boxed{x \neq p}$

$$\mu(x) = \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p)^m q'(x)} =$$

$$= \frac{(x-p)^{m-1}}{(x-p)^{m-1}} \frac{(x-p)\, q(x)}{m\, q(x) + (x-p)\, q'(x)}$$

<u>Definition.-</u>

First consider that if "p" is a zero of multiplicity m, then

$$f(x) = (x-p)^m q(x)$$

and $\lim_{x \to p} q(x) \neq 0$. Also, $f(p) = 0 = f'(p) = \dots f^{(m-1)}(p)$ but, $f^{(m)}(p) \neq 0$

Therefore, if we define

$$\boxed{\mu(x) = \frac{f(x)}{f'(x)}, \qquad x \neq p}$$

$$\mu(x) = \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p)^m q'(x)}$$

or

$$\boxed{\mu(x) = \frac{(x-p)\, q(x)}{m\, q(x) + (x-p)\, q'(x)}} \qquad (*)$$

From the last formula$^{(*)}$ for $\mu(x)$, we see that $\mu(p) = 0$ is defined

Thus, if we define

$$\mu(x) = \begin{cases} \dfrac{f(x)}{f'(x)} = \dfrac{(x-p)\, q(x)}{m\, q(x) + (x-p)\, q'(x)}, & x \neq p \\ 0, & x = p. \end{cases}$$

we will have a continuous and a smooth function at $x = p$.

Also,

$$\mu'(x) = \frac{[q + (x-p) q'][mq + (x-p) q'] - [mq' + q' + (x-p) q''] * (x-p) q}{[mq + (x-p) q']^2}$$

and $$\mu'(p) = \frac{m q^2(p)}{m^2 q^2(p)} = \frac{1}{m} \neq 0.$$

Therefore, $\mu(x)$ has a simple root at $x = p$ and Newton iteration should converge to p of order 2.

The functional iteration for the new function $\mu(x)$ is obtained from the function $f(x)$.

In fact,

$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{\frac{f(x)}{f'(x)}}{\left[\frac{f(x)}{f'(x)}\right]'}, \quad x \neq p.$$

$$= x - \frac{f(x)/f'(x)}{\left[f'^2(x) - f f''(x)\right] / \left[f'(x)\right]^2}$$

$\therefore$

$$\boxed{g(x) = x - \frac{f(x)\,f'(x)}{f'^2(x) - f(x)\,f''(x)}}$$

Functional iteration

$$\boxed{p_{n+1} = g(p_n) = p_n - \frac{f(p_n)\,f'(p_n)}{\left[f'(p_n)\right]^2 - f(p_n)\,f''(p_n)}} \quad (**)$$

Its application requires knowledge of $f(x)$, $f'(x)$ and $f''(x)$.

Back to exercise 1a Sect 2.4 Burden.

$$f(x) = x^2 - 2xe^{-x} + e^{-2x}$$

$$f'(x) = 2\left[-e^{-2x} + xe^{-x} - e^{-x} + x\right]$$

$$f''(x) = 2\left[2e^{-2x} - xe^{-x} + 2e^{-x} + 1\right].$$

Execute MATLAB CODE.

Compare with Newton iteration.

18


```
>>    Input data
     A              B           TOL          itmax          po
          0  1.4000e+000  1.0000e-010  2.0000e+002  2.5000e-001

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4.0000e+000	5.4488e-001	2.1909e-002	1.2276e-003	NaN
5.0000e+000	5.5596e-001	1.1088e-002	3.0815e-004	NaN
6.0000e+000	5.6154e-001	5.5780e-003	7.7194e-005	NaN
7.0000e+000	5.6434e-001	2.7975e-003	1.9318e-005	NaN
8.0000e+000	5.6574e-001	1.4009e-003	4.8320e-006	NaN
9.0000e+000	5.6644e-001	7.0098e-004	1.2083e-006	NaN
1.0000e+001	5.6679e-001	3.5062e-004	3.0212e-007	NaN
1.1000e+001	5.6697e-001	1.7534e-004	7.5534e-008	NaN
1.2000e+001	5.6706e-001	8.7681e-005	1.8884e-008	NaN
1.3000e+001	5.6710e-001	4.3842e-005	4.7211e-009	NaN
1.4000e+001	5.6712e-001	2.1922e-005	1.1803e-009	NaN
1.5000e+001	5.6713e-001	1.0961e-005	2.9507e-010	NaN
1.6000e+001	5.6714e-001	5.4805e-006	7.3768e-011	NaN
1.7000e+001	5.6714e-001	2.7403e-006	1.8442e-011	NaN
1.8000e+001	5.6714e-001	1.3701e-006	4.6105e-012	NaN
1.9000e+001	5.6714e-001	6.8507e-007	1.1526e-012	NaN
2.0000e+001	5.6714e-001	3.4254e-007	2.8810e-013	NaN
2.1000e+001	5.6714e-001	1.7124e-007	7.2109e-014	NaN
2.2000e+001	5.6714e-001	8.5704e-008	1.8041e-014	NaN
2.3000e+001	5.6714e-001	4.2914e-008	4.4409e-015	NaN
2.4000e+001	5.6714e-001	2.1187e-008	1.2212e-015	NaN
2.5000e+001	5.6714e-001	1.1571e-008	2.7756e-016	NaN
2.6000e+001	5.6714e-001	5.6983e-009	1.1102e-016	NaN
2.7000e+001	5.6714e-001	5.3586e-009	0	NaN

```
pn is an exact root
Exact Root and Number of Iterations
    5.671432915503317e-001    2.700000000000000e+001
```

18

```
>>     DATOS DE ENTRADA
         TOL             itmax             po
   9.999999999999999e-006    2.000000000000000e+002    5.000000000000000e-001

         it          aprox.         error        f(pn)
   1.0000e+000  5.6801e-001  6.8014e-002  1.8602e-006

   2.0000e+000  5.6714e-001  8.7031e-004  4.6019e-014

    Final Results
    it            aprox.         error              f(pn)
   3.0000e+000  5.6714e-001  1.3659e-007              0

>>     DATOS DE ENTRADA
         TOL           itmax           po
   1.0000e-010  2.0000e+002  5.0000e-001

         it          aprox.         error        f(pn)
   1.0000e+000  5.6801e-001  6.8014e-002  1.8602e-006

   2.0000e+000  5.6714e-001  8.7031e-004  4.6019e-014

   3.0000e+000  5.6714e-001  1.3659e-007            0

    Final Results
    it            aprox.         error              f(pn)
   4.0000e+000  5.6714e-001            0              0

>>
```